Linear and Partial Orderings

by Sidney Felder

When we are given the task of ordering a class of objects, whether concrete or abstract, it tends to be in a context in which it is appropriate to place them in an ordering that corresponds naturally to the arrangement of objects in a row or line. Comparisons of natural numbers according to greater and less, comparisons of length, comparisons of temperatures (say, in degrees Celsius), and comparisons of intervals of time since the origin of the Universe, are all examples of *Linear Orderings* (also called *Total*, *Simple*, or *Serial* Orderings). Linear orderings, like all orderings, possess an *underlying domain* on which the ordering is defined. (In the cases mentioned above, the underlying domains are, respectively, the set of natural numbers; a set of linear extensive magnitudes; a set of temperature measurements; and a set of time intervals all beginning at the same instant). Linear orderings, as distinct from various weaker orderings that are also extremely important, have the property that given any two elements of the *underlying domain* x and y (not necessarily distinct), either x precedes y in the ordering, x occupies the same position as y in the ordering, or y precedes x in the ordering. When these conditions are met, we say that any two elements of the underlying domain are *mutually comparable*, or that all the elements of the domain are *connected*.

The notation normally used to represent the generic relation of order precedence is the standard symbol for 'less than', <, which represents numerical and quantitative order in the corresponding more specialized domains. Thus when employed in the context of the consideration of orderings, x<y is translated 'x precedes y in the ordering', x=y is translated 'x occupies the same position as y in the ordering', and x>y is translated 'y precedes x in the ordering' (or equivalently, 'x succeeds y in the ordering').[1] It is very common, and frequently more useful, to employ the relation 'x does not succeed y in the ordering' (equivalent to the disjunction of the relations 'x precedes y' and 'x occupies the same position as y') as the basic relation, in which case the 'less-than-or-equal' sign ≤ is typically used in place of <. There is some variation in the convention for '<', but when it is said "x *strictly precedes* y", the possibility that x and y occupy the same position in the ordering is ruled out. (I should take the opportunity to emphasize that when we say simply that A is a subset of B, we never wish to exclude the possibility that A corresponds to the whole of B: When we wish to exclude this latter possibility, we say that A is a *proper subset* of B). An ordering is typically defined by an ordered pair whose first term represents the underlying domain of the ordering, and whose second term indicates the respect in which the domain is to be ordered. Thus, for example, (*N*,numerical order) defines the ordering of the natural numbers by numerical magnitude.

A *Partial Ordering* is the ordering relation that one obtains if one drops the condition that any two elements of the underlying domain be comparable. (Again, in saying that a relation on a set forms a partial ordering, we do not mean to exclude the possibility that the partial ordering is a linear ordering: In other words, the class of partial orderings is defined to encompass the class of linear orderings). Thus consider the following orderings: 1) sets ordered by the relation of inclusion ($\subseteq$, not $\in$); 2) the set of natural numbers (excluding 0), N-{0}, ordered by the relation of divisibility

[1] In the cases in which we are interested here, any two elements that occupy the same position are identical. (An ordering that possesses this property is called *anti-symmetric*). This contrasts with the kind of structure called a *pre-ordering*, in which we permit a multitude of distinct elements to occupy the same position. The ordering of points in space by their distances from the center of the Earth is a pre-ordering because for each specified non-zero distance from the center of the Earth, there exists a whole spherical surface all of whose (infinite number of) points possess the specified distance.

without remainder; 3) the set of positions on Earth ordered by magnitude of latitude and longitude coordinates; and 4) the set of blood pressure measurements ordered by numerical readings (such as ‘120 over 80’). None of these, as they stand, are linear orderings. According to the first formally simple definition that comes to mind “(a,b)<(c,d) if a<c and b<d”, the blood pressure ‘200 over 100’ is higher than the blood pressure ‘190 over 70’ and higher than the blood pressure ‘120 over 80’, but the latter two blood pressures are, strictly speaking, incomparable. (Indeed, we tend to call the second blood pressure ‘high’ and the last ‘normal’ because the very substantial elevation of the systolic blood pressure is a central feature and cause of many common conditions of unhealth). Similarly, any two cities are comparable 1) in relation to their degrees of elevation from sea level, 2) in relation to their latitudes, and 3) in relation to their longitudes, but we cannot compare Moscow (latitude 55 N, longitude 37 E) with Tokyo (latitude 35 N, longitude 139 E) in relation to latitude-longitude coordinates and say which is “greater”. Of course, we can always *extend* the “natural” partial ordering on latitude-longitude pairs to a linear ordering by imposing some arbitrary order of precedence—one of course that *preserves* (i.e., that is *consistent* with) the original formally “natural” partial ordering—on all latitude-longitude pairs, whether intuitively comparable or not. However, it is not to be expected that the extended scheme that results from interposing these intuitively mutually incomparable latitude-longitude pairs will have any organic connection to the relation defining the partial ordering, let alone to the two separate relations of comparative latitude and comparative longitude that define the component linear orderings.

In both the medical and geographical examples discussed above, the basic structural fact is that two distinct attributes are involved, and there is no particular combination of them that corresponds to anything meaningful. Is it appropriate to add longtitude and latitude measures? What about multiplying them? The lack of significance that attaches to any definable partial or linear ordering on latitude-longitude pairs stands in marked contrast to the familiar natural Pythagorean combination of coordinate differences along mutually orthogonal axes in space that defines *distance*. (A point that is 3 miles to the east of the Empire State Building and 4 miles to the north of the Empire State Building is five miles distant from the Empire State Building).[2] Intrinsic to the mathematical structures of examples (1) and (2) above, on the other hand, are non-total partial ordering relations of great significance and fertility.

We first consider the set of natural numbers (excluding 0), *N*-{0}, ordered by the relation of divisibility without remainder. (In this context, ‘divisibility’ always means ‘divisibility without remainder’). Consider any number, say 3. By the definition of (2), any multiple of 3 is comparable to it, though there are clearly multiples of 3—6 and 9 for example—that are not comparable to each other (6 and 9 are both divisible by 3, but neither 6 nor 9 is divisible without remainder by the other). Any subset of mutually comparable elements of a set (according to the specified ordering relation) is called a *chain*. A chain, in other words, is a linearly ordered subset of a partially ordered set. The subset of powers of 2 forms an infinite set {2,4,8,16,32,64,...} all of whose elements are mutually comparable. (Given any pair of powers of 2, m and n, either m divides n without remainder, or n divides m without remainder). This subset also happens to form a *maximal chain* in *N*, meaning that there is no natural number outside it that is comparable to *every* number belonging to it. (Although the number 160 is comparable to the numbers 2, 4, 8, 16, and 32, it is not an element of

[2] According to the Special Theory of Relativity, space-time forms a *4-dimensional continuum*, not in the obvious sense that it requires three spatial and one temporal coordinate to unambiguously locate each event in space and time, but in the deeper sense that a certain combination of spatial and temporal coordinate differences (the *Interval*) possesses greater physical significance than does any purely spatial combination of coordinate differences.

this chain because there is some element in it (64) by which it is not divisible). In other words, there is no proper superset of this set—no subset of *N*-{0} containing all powers of 2 and some other numbers besides—all of whose elements are mutually comparable.

There are also sets *none* of whose elements are mutually comparable; these "cross-sections" of chains are called, appropriately enough, *anti-chains*. Any set of prime numbers (numbers that are divisible only by themselves and 1) forms an anti-chain. The infinite set of *all* prime numbers (2,3,5,7,11,13,17,...) is a *maximal anti-chain*, meaning that there is no number outside this set that is *incomparable* to *all* numbers within it. (This is because every number that is not prime is *composite*, a product of primes. Consequently, each number not belonging to the set of primes is divisible without remainder by some element belonging to the set of primes). Note well that there an infinite number of maximal chains (the powers of 3, the powers of 5, the powers of 7,... all form maximal chains). There are also an infinite number of maximal anti-chains (the set of all the prime numbers to the second power, the set of all the prime numbers to the third power, etc. all form maximal anti-chains) for this ordering relation. (All maximal chains and anti-chains for this domain and ordering relation contain an infinite number of elements). As can easily be verified, maximal chains may or may not overlap.

The partial ordering of sets by the relation of inclusion ('A is a subset of B') is one of the two maximally important partial ordering relations; it also provides some of the clearest illustrations. (The other partial ordering relation I have in mind is that of *logical implication*). Consider the set {A,B,C,D}. The set of all its subsets, what is called the *power set* of {A,B,C,D}, contains the following sixteen sets: the null set { } (the set containing no elements), {A}, {B}, {C}, {D}, {A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}, {A,B,C}, {A,B,D}, {A,C,D}, {B,C,D}, and {A,B,C,D} itself. (Remember that the null set is considered the subset of *every* set, and that a set is always counted as a subset of itself). We now look at the partial ordering of this set of subsets by the relation of inclusion. The three sets {A}, {A,B}, and {A,B,C,D} are all mutually comparable, and hence their combination forms a chain; however, they do not form a maximal chain—the null set and the set {A,B,C} are both comparable to {A}, {A,B}, and {A,B,C,D}. All five of these sets are mutually comparable (by the relation of inclusion), and no *other* set is comparable to all five, so the class of these five sets is a maximal chain. There are a multiplicity of distinct chains in the power set of {A,B,C,D}, and also a multiplicity of maximal chains, four to be exact. (Although in this example all maximal chains happen to contain the same number of elements (5), this does not in general have to be the case).

This fact highlights the distinction between the concepts of *maximum* and *greatest*, a distinction that is central to the characteristic logic of partial orderings. Consider any finite set of natural numbers in relation to their numerical order. There is obviously necessarily both a greatest number in this set—a number greater than any other number in this set—and a least number in this set—a number smaller than any other number in this set. Now consider the twelve element set {{A,B,C}, {A,B,D}, {A,C,D}, {A,B}, {C,D}, {A,D}, {B,C}, {A}, {B}, {E,F}, {G}, {H,I,J,K}}. This finite set contains no element that precedes every other element in the inclusion ordering (i.e. no set that is included in every other one of the eight sets), and no element that succeeds every other element in the inclusion ordering (i.e., no set that includes every other one of the eight sets). It does possess what are called *minimal* and *maximal* elements, the former being elements that are preceded by no other elements, and the latter being elements that are succeeded by no other elements. (The minimal elements of this set are {A}, {B}, {C,D}, {E,F}, {G}, and {H,I,J,K}. The maximal elements of this set are {A,B,C}, {A,B,D}, {A,C,D}, {E,F}, {G}, and {H,I,J,K}. There is no least element in this set. (The convention that ∅ is a subset of every set does not mean that ∅ is an *element* of every set). There is also no greatest element of this set). Alternatively stated, the minimal and maximal elements of a set are the least and greatest elements respectively of a partially ordered set's maximal chains. Or, perhaps more simply, a minimal element of a set is the least of all the elements that are comparable to *it* with respect to the ordering relation in question. (A linear ordering on a set possesses at most one minimal element and at most one maximal element, and (when these exist) they correspond to the least and greatest elements of the set respectively). Note that the same element may be both a minimal and a maximal element of a set with respect to the given ordering relation however numerous the set, and that there can be a multiplicity of such simultaneously minimal and maximal elements. On the other hand, an element can be both the greatest and the least element of a set only if it is the sole element of the set.

The symmetrical concepts of *least upper bound* and *greatest lower bound* are of great logical and mathematical significance. Consider again a finite set of natural numbers in order of magnitude, stretching from the number 20 to the number 18,000. (Whether or not this set includes all the numbers between 20 and 18,000 is immaterial). An *upper bound* for an ordered set is an element that is either greater or equal to each element in that set. In this case, 20,000 is an upper bound, as is every number equal to or greater than 18,000. Among all these upper bounds, there is a least, and this, naturally enough, is called the *least upper bound* (*supremum* or *lub* for short) of the set. The analogous construction exists for lower bounds. In our example, the numbers 0 to 20 are lower bounds to the set, but 20 is the *greatest lower bound* (*infimum* or *glb* for short). (Notice the way 'greatest' is coupled with 'lower' and 'least' is coupled with 'upper'). The notions of least upper bound and greatest lower bound, at least so far as linear orderings are concerned, only become interesting when *infinite* bounded sets are involved. Consider now the *closed set* of all *rational numbers* between 1 and 2 inclusive [1,2]. (A rational number is one that can be expressed as a "fraction"—a ratio of whole numbers). This is a set with an infinite number of elements, but one that is bounded in the geometrical sense, that is, entirely confined to a concentrated region on the number line. (A set of points on a space is *bounded* if there exists (i.e., if it is possible to specify) a finite degree of separation that is greater than the separation between any two points in the set, that is, if there is a largest finite separation between any chosen point of the set and *every* other point in the set). The least upper bound of this set is the number 2, the greatest lower bound is 1, and both these elements belong to the set itself. On the other hand, consider the *open set* of rational numbers between 1 and 2 *exclusive*, the same set as before except without the boundary points. The least upper bound of this set is again 2, but this element is *outside* the set: There is no upper bound to this set that lies within it.

Least upper bounds and greatest lower bounds also appear in the context of partial orderings, and in a particularly natural way in the ordering relations of set inclusion and logical implication. In the partial ordering of natural numbers generated by the relation of divisibility without remainder, the least upper bound of any two numbers x and y is the old familiar *Least Common Multiple* of x and y, the least number that is divisible by both x and y individually. The product of x and y is always *an* upper bound of x and y in relation to divisibility, but it is not typically the *least* upper bound. Thus although 8 and 12 both divide the number 96 (the product of 8 and 12), the least upper bound of 8 and 12 is 24, the smallest number that is divisible by both 8 and 12. The greatest lower bound of x and y is the equally well-known *Greatest Common Divisor* (or *Greatest Common Factor*) of x and y, the greatest number that divides both x and y.

The set of *all* subsets of any set ordered by inclusion forms a special kind of partial ordering, called a *lattice*, which can be defined as a partially ordered set in which any two elements have both a least upper bound and a greatest lower bound.[3] An upper bound of a pair of sets X and Y is a set that contains every element belonging to either X or Y. The least upper bound of X and Y is the least comprehensive of these sets, which is the intersection of all the upper bounds of X and Y. This intersection of all upper bounds corresponds to the *union* of X and Y—the set of elements that belong to at least one of the two sets. A lower bound of X and Y is a set that is contained in both X and Y. Among these sets the greatest lower bound is the most comprehensive, and is the set corresponding to the union of all the lower bounds. This union of all lower bounds corresponds to the *intersection* of X and Y.

As we will see, this pattern carries over essentially unchanged to the relation of implication among formulae of propositional logic, with conjunction taking the place of intersection, and disjunction taking the place of union. Thus given four propositions A, B, C, and D, the greatest lower bound of A and B in relation to the ordering by logical strength is A&B; it is the weakest of all the combinations of propositions A&B, A&B&C, A&B&C&D that imply both A *and* B (A&B is in fact equivalent to the *disjunction* of A&B, A&B&C, A&B&C&D). Similarly, $A \vee B$ is the least upper bound of A and B in relation to this ordering, being the strongest formula that is implied by *both* A and B separately. A implies $A \vee B$, $A \vee B \vee C$, and $A \vee B \vee C \vee D$ *and* B implies $A \vee B$, $A \vee B \vee C$, and $A \vee B \vee C \vee D$; $A \vee B$ is the *conjunction* of $A \vee B$, $A \vee B \vee C$, and $A \vee B \vee C \vee D$.

I would like to briefly introduce a few particularly important linear orders.

A *discrete ordering* is an ordering, or more properly speaking, an *order-type*, in which all but at most two elements possess both an immediate successor and an immediate predecessor. Such an ordering permits us to speak of *the* next element and *the* previous element. In other words, it is an order in which any two elements, whether immediately *contiguous* (*adjacent*) or not, are separated by only a finite number of other elements. The least element, if it exists, has one immediate successor and no predecessor, and the greatest element, if it exists, has one immediate predecessor and no successor.

A *finite discrete ordering* is an ordering possessing both an element without a predecessor (the first element) and an element without a successor (the last element), where each element of the ordering except the first possesses exactly one immediate predecessor, and each element of the ordering except the last possesses exactly one immediate successor. The n-tuples (0,1,2,3) and (a,b,c,d,e) are examples.

An *infinite progression* (the modifier ‘infinite’ is sometimes omitted) is a discrete ordering that possesses a first element and *no* last element, and in which every element but the first possesses exactly one immediate predecessor, and *every* element has an immediate successor. The infinite series (0,1,2,3,...)—the set of natural numbers in order of magnitude—is the prototypical example of a progression. Other examples are (17,18,19,20,21,...), (0,5,10,15,20,...), (2,3,5,7,11,13,...), and (1/2,1/4,1/8,1/16,...). (Progressions are said to have the *order-type* ω. Unfortunately, the symbol ω is conventionally used both to represent the particular *set* of natural numbers {0,1,2,3,...} and the

[3] For finite sets, this condition is equivalent to the (in general) stronger condition that every subset of elements has both a supremum and an infimum. Any set and ordering relation that satisfies the stronger condition is called a *complete lattice*. While all finite lattices are complete lattices, there are lattices with infinite numbers of elements that do not form complete lattices.

general *order-type* of an infinite progression).

An *infinite regression* is a discrete ordering that possesses a *last* element and no first element, and in which every element but the last possesses an immediate successor, and *every* element possesses exactly one immediate predecessor. Such regressions are said to have the order-type $*\omega$ (sometimes written $\omega*$). The infinite series (...,-3,-2,-1,0), in which the non-positive integers are arranged in order of magnitude, is the prototypical example. Note that we can instead arrange the *set* of non-positive integers in order of their *absolute values*, producing the series (0,-1,-2,-3,...), an infinite progression. (The absolute value of an integer z is equal to z if z is non-negative and equal to -z if z is negative). Similarly, we can arrange the non-positive integers in inverse order of magnitude, producing the series (...,3,2,1,0), an infinite regression. In general, any set containing two or more elements can be ordered in more than one way, and any infinite set can be ordered in an infinite number of ways.

The infinite progression ω is a special case of a family of order-types called *well-ordered series*. Such series possess a first element and an immediate successor to every element that belongs to the series. The series (0,2,4,6,...1,3,5,7...), the progression of even numbers followed by the progression of odd numbers, is a well-ordered series corresponding to the order-type $\omega+\omega$. The numbers 1, 3, 5, 7 etc. are all conceived to occupy positions in the series that follow *all* the even numbers. The first odd number, the number 1, is the only element of the series besides the first element 0 that has no immediate predecessor: Given any even number *2n* preceding 1, there is another even number (2n+2) in the series that occupies a position between *2n* and *1*. On the other hand, every element of the series possesses a unique immediate successor. The series (0,3,6,9,...1,4,7,10,...2,5,8,11,...) possesses the order-type $\omega+\omega+\omega$, the numbers 0, 1, and 2 being the only elements without an immediate predecessor. Taking the series of series further, we obtain $\omega+\omega+\omega+\omega$, $\omega+\omega+\omega+\omega+\omega$,... until, by continued thinnings and concatenations, we reach an infinite series that contains an infinite number of progressions, usually called ω^2. Speaking now not merely about the number of terms but of the number of distinct order types that can be generated, this is just the beginning, and there is no end.[4] In general terms, a well-ordered series is a series in which *every* subset, however dispersed its elements, possesses a first element with respect to the underlying ordering relation. (Another way to state this last condition is that well-ordered series contain no infinite regressions).

A *densely ordered series* is a series in which for any two of its elements, there is another element that lies between them. The most familiar example is *Q*, the set of *rational numbers* ordered by magnitude. (A rational number is a number whose magnitude can be expressed in the form of a simple fraction a/b, where *a* and *b* are natural numbers, *b* not equal to 0. Note that any two fractions such as a/b and 2a/2b that have the same numerical ratio are considered to be the same rational number, which implies that each rational number corresponds to an infinite number of fractions). It can easily be demonstrated that *Q* arranged in order of magnitude forms a dense series: Given any two rational numbers a/b and c/d, there exists another rational number (ad+cb/2bd), their average,

[4] We should now make explicit the distinction between an ordering and the more abstract notion of an *order-type*. Consider the set of elements {a,b,c}. There are six possible ways to (linearly) order this set (a,b,c), (b,c,a), (a,c,b), etc. However, all of these are three-element discrete orderings, and are thus considered to have the same order-type. Similarly, the orderings of the natural numbers (0,1,2,3,4,5,...) and (1,0,3,2,5,4,...) are different orderings of the set of natural numbers, but both are simple infinite progressions. There are obviously an infinite number of progressions that can be formed from the set of natural numbers. And, as should be obvious from the paragraph above, it is also the case that the set of natural numbers {0,1,2,3,...} can be arranged in an infinite number of distinct order-types.

that lies between a/b and c/d. This implies, in fact, that there are an infinite number of rational numbers between any two rational numbers. Interestingly, there is a well known method for ordering the rational numbers in an infinite progression. Such a progression, of course, does not order the rationals in order of magnitude.

It was demonstrated thousands of years ago (definitely by the Pythagoreans about 2500 years ago and possibly in other civilizations) that there are linear magnitudes that cannot be expressed as whole number ratios of other magnitudes. (In other words, there exist pairs of magnitudes *m* and *n* that cannot both be expressed as multiples of the same smaller magnitude). Such magnitudes are called *incommensurable*. This immediately implies that there are points on the real line whose positions are not expressible as rational numbers, and hence that there are gaps on the number line, positions not occupied by any rational number. This geometrical fact can be rendered in algebraic terms by the statement that the system of rational numbers arranged in order of magnitude is *incomplete*. To get a sense of this notion, consider the set of rational numbers greater than 1 and less than the square root of 2 (the square root of 2 is an *irrational* number), conceived as embedded in the system of rational numbers. Unlike any open or closed interval whose bounds are rational numbers, this set has no least upper bound in the system of rational numbers at all, whether inside or outside this bounded set. This is one of the reasons that the expansion of the set of points on the number line beyond the rational numbers to encompass the irrational numbers is considered desirable by most mathematicians. At least within the framework of classical mathematics, it can be proven that every bounded subset of the *continuum* (in algebraic terms, the system of real numbers) has both a least upper bound and a greatest lower bound either inside or outside the bounded subset.

Finally, we should say something about *Archimedean* and *non-Archimedean* orderings. In an ordinary algebraic and geometric context, a system is Archimedean if given any pair of magnitudes s and t, it is always possible to find some integer r such that rs>t. In the system of real numbers ordered by numerical magnitude, this means that no matter how large a number t we choose, and how small a number s we choose, there is some integer r such that the product of r and s is greater than t. In a geometric setting, this translates into the claim that whatever two lengths s and t we choose, there is some integer r such that the product of r and the length s of the shorter line segment exceeds the length t of the longer line segment.

The simplest non-Archimedean linear ordering most people are familiar with is alphabetical order (in more general contexts, referred to as *lexicographic order*). This is the order of words in a dictionary and of names in a phone book. The words in a dictionary are (linearly) ordered as follows: First, *all* words whose first letter is ‘a’ are placed ahead of *all* words whose first letter is ‘b’; all words whose first letter is ‘b’ precede all words whose first letter is ‘c’; etc., until we reach the letter ‘z’. This leaves all words with the same first letter completely unordered. We then order the set of words whose first letter is ‘a’ by placing all words whose second letter is ‘a’ ahead of all words whose second letter is ‘b’; all words whose second letter is ‘b’ are placed ahead of all words whose third letter is ‘c’, etc. This leaves all words with the same first and second letters completely unordered. We remedy this by arranging all words that begin ‘aa’ in accordance with the alphabetical order of their third letters; etc. The idea is that the second, third, and fourth letters of a word are not even considered until the words are sorted by first letters. In other words, later letters in words count for nothing, have no weight at all, in comparison with earlier letters in words: Any string of letters that begins with ‘aa’, even if these are the first letters of the string ‘aazzzzzzzzzz’, must precede abaaaaaaaaaa’. No amount of ‘z’s’ added to a string beginning with ‘aa’, or of ‘a’s’ added to a string beginning with ‘ab’, can ever move a string beginning ‘ab’ ahead of a string beginning with ‘aa’. At this point, the connection of lexicographic and non-Archimedean ordering should be clear. (Non-Archimedean orderings are implicit in the (now demonstrably coherent (Abraham Robinson)) language of infinitesmals; they are also believed by some to play an important role in ethics and decision theory).